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Conjugate asymptotic properties of spectra and correlations

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Abstract. We examine a general technique for deriving the small time asymptotic expansion of a correlation from the large frequency asymptotic form of the associated spectrum (*conjugate* asymptotic properties). Our analysis explicitly takes into account the form of approach of the spectrum to its asymptotic limit (i.e. asymptotic convergence), and the resulting impact on the correlation asymptotic expansion. We fully evaluate the two lowest-order terms in the small time asymptotic expansion of the correlation for the important special case of the large frequency asymptotic behaviour of the spectrum being a negative power of frequency. Included in our analysis is a determination of sufficient conditions on the rapidity of approach of the spectrum towards its asymptotic form (i.e. convergence rate), for the derived correlation asymptotic approximation to be accurate to second order. We comment on how small time must be for our correlation asymptotic approximations to be valid. To motivate this analysis we propose circumstances under which these results could be of utility in physics.

1. Introduction

The conjugate concepts of *spectrum* $S(\omega)$ and *correlation* $r(t)$ arise quite naturally and commonly in many aspects of physics that have a probabilistic element in their specification or description. Natural phenomena often are characterized by spectra or correlations that asymptotically decay with a negative power law as ω or t increases without bound.

A selection of such examples emanating from recent research is: temporal autocorrelations of electrons in random potentials and random magnetic fields [1]; temporal autocorrelations of electrons in random potentials with spin–orbit interactions [2]; vortex orbit radius temporal autocorrelations for turbulent flow [3]; spatial autocorrelations of airborne infrared imagery of natural terrain [4]; various spatial correlations and spectra occurring in critical phenomena [5]; surface roughness spatial spectra [6]; atmospheric airglow spatial spectra [7]; surface gravity wave height spatial spectra [8]; and fluorescence temporal spectra [9]. In addition, power spectra that strictly obey a negative power law over a very large domain of frequencies feature prominently in the theory of fractal self-similarity [10], and such power spectra are the foundation of the theory of $1/f$-noise [11], and its generalization to coloured noise [12].

The research on theoretical and experimental investigations of surface roughness reported by Yang and Lu [6] is especially conducive to application of the specific conjugate asymptotic relations derived in section 5. Yang and Lu's analysis features frequent conversions between spectrum asymptotic behaviour towards infinity and correlation asymptotic behaviour near zero, which is the problem that we shall address here.

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We shall introduce and define the mathematical meaning of correlations and spectra in section 2. Our analysis will proceed from the basis that $S(\omega)$ somehow approaches a certain functional form as $\omega \to \infty$. Section 3 makes mathematically precise the meaning of 'somehow approaches' by invoking the concept of asymptotic convergence. The asymptotic expansion of $r(t)$ as $t \to 0$, given the conjugate asymptotic behaviour of $S(\omega)$ as $\omega \to \infty$, is derived to the extent possible while maintaining full generality in section 4. For the important special case of $S(\omega)$ approaching a power-law decay at large ω, the two lowest-order terms of the small t asymptotic expansion of $r(t)$ are explicitly derived in section 5, for all possible powers of $S(\omega)$ decay. The analysis of section 5 includes a derivation of sufficient conditions on the rapidity with which $S(\omega)$ approaches its asymptotic functional form as ω increases, for the derived small t asymptotic approximations of $r(t)$ to be accurate to second order. Section 6 explains and reinforces the fact that the derived asymptotic convergence conditions on $S(\omega)$ are sufficient, but not necessary, conditions for assurance of the precision of the derived $r(t)$ asymptotic approximation. As motivation for this work, we conclude in section 7 with general examples of physics research for which our results could be of considerable benefit.

The present analysis pertains to the $\omega \to \infty$ spectrum behaviour implication for the $t \to 0$ correlation behaviour. Converse results, that is, the $t \to 0$ correlation behaviour implication for the $\omega \to \infty$ spectrum behaviour are accessible from this analysis, but we will not extend the analysis to their derivation, since these latter results are also accessible from the generalized function analysis of Lighthill [13]. The present results are special cases of particular interest to physics, within the general field of asymptotic approximation [14], and their analysis and elucidation has attracted some recent attention [15–17].

2. Correlations and spectra

The *autocorrelation* of a real, stationary stochastic process $\boldsymbol{x}(t)$ (bold type indicates a random variable) is defined as

$$r(t) \equiv \langle \boldsymbol{x}(s)\boldsymbol{x}(s+t)\rangle \tag{1}$$

where $\langle\cdot\rangle$ indicates the operation of mathematical expectation or ensemble averaging, and s is an arbitrary time value. The *power spectrum* of $\boldsymbol{x}(t)$ is defined as the Fourier transform of $r(t)$, that is,

$$S(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} r(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{2}$$

from which it follows that $r(t)$ is regained from $S(\omega)$ by the inverse Fourier transform

$$r(t) = \int_{-\infty}^{\infty} S(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega. \tag{3}$$

$r(t)$ is real and even, so it follows that $S(\omega)$ is both real and even; accordingly (3) simplifies to

$$r(t) = 2\int_{0}^{\infty} S(\omega)\cos\omega t\,\mathrm{d}\omega. \tag{4}$$

Likewise, (2) simplifies to

$$S(\omega) = \frac{1}{2\pi}2\int_{0}^{\infty} r(t)\cos\omega t\,\mathrm{d}t. \tag{5}$$

The identicality, to within a constant factor, of the reciprocal relations (4) and (5) implies that conjugate pairs of asymptotic relations are necessarily symmetric in the following sense.

If we swap t with ω, and rename the new function of t: $r(t)$, and the new function of ω: $S(\omega)$, then we always have the correct dual conjugate pair of asymptotic relations, to within a constant multiplier. In essence, if we make the effort to discover that a particular large frequency behaviour of $S(\omega)$ implies a particular small time behaviour of $r(t)$, then by symmetry we know also that the analogous large time behaviour of $r(t)$ implies the analogous small frequency behaviour of $S(\omega)$.

3. Asymptotic convergence

Let us define precisely what is meant by the intuitive notion that $S(\omega)$ asymptotically approaches the positive semidefinite function $S_{\mathrm{as}}(\omega)$ as $\omega \to \infty$, which is symbolically written as $S(\omega) \sim S_{\mathrm{as}}(\omega)$ as $\omega \to \infty$.

$S(\omega)$ will be defined to *asymptotically converge* upon $S_{\mathrm{as}}(\omega)$ as $\omega \to \infty$ with positive definite *convergence function* $C(\omega)$, if for every $\varepsilon_0 > 0$ there exists an $\omega_0 > 0$, such that $|S(\omega) - S_{\mathrm{as}}(\omega)| < \varepsilon_0 C(\omega)$ for all $\omega > \omega_0$. This criterion may be succinctly expressed by use of the Landau $o(\cdot)$ notation:

$$S(\omega) \sim S_{\mathrm{as}}(\omega) \text{ as } \omega \to \infty \equiv |S(\omega) - S_{\mathrm{as}}(\omega)| = o(C(\omega)) \qquad \text{as } \omega \to \infty. \tag{6}$$

In words, $S(\omega)$ asymptotically converges upon $S_{\mathrm{as}}(\omega)$ as $\omega \to \infty$, if the discrepancy between $S(\omega)$ and $S_{\mathrm{as}}(\omega)$ decays away faster than a certain $C(\omega)$ as ω increases without bound. Sensible choices of $C(\omega)$ monotonically decay to zero as ω increases.

Note that for given $S(\omega)$ and $S_{\mathrm{as}}(\omega)$ there is an innumerable infinity of valid convergence functions. For example, if one particular $C(\omega)$ is known to be valid, then so is any function that is at least as large as the original $C(\omega)$ everywhere beyond a certain value of ω. However, not all possible convergence functions are equally useful, for the following reason. The following analysis will impose sufficient conditions on the 'smallness' of $C(\omega)$ for the derived asymptotic approximations to be valid. Therefore, there is a benefit in identifying a $C(\omega)$ that is as small as possible (to within a constant multiple); that benefit being that there is then a maximal likelihood that the identified $C(\omega)$ is sufficiently small to guarantee the validity of the asymptotic approximation to the implied precision. If our convergence function is not sufficiently small, then we remain uncertain about whether our asymptotic approximation is as accurate as we expect.

4. General asymptotic analysis

On choosing an arbitrarily small positive ε_0, and a sufficiently large ω_0 according to the asymptotic convergence criterion of section 3, we expand (4) into

$$r(t) = 2\int_0^{\omega_0} S(\omega)\cos\omega t \,\mathrm{d}\omega + 2\int_{\omega_0}^{\infty} S(\omega)\cos\omega t \,\mathrm{d}\omega \tag{7}$$

which may be expressed in terms of $S_{\mathrm{as}}(\omega)$ as

$$r(t) = 2\int_0^{\omega_0} S(\omega)\cos\omega t \,\mathrm{d}\omega + 2\int_{\omega_0}^{\infty} S_{\mathrm{as}}(\omega)\cos\omega t \,\mathrm{d}\omega + R(t) \tag{8}$$

where the *remainder* $R(t)$ is defined as

$$R(t) \equiv 2\int_{\omega_0}^{\infty} (S(\omega) - S_{\mathrm{as}}(\omega))\cos\omega t \,\mathrm{d}\omega. \tag{9}$$

Using the asymptotic convergence criterion of section 3, we obtain an upper bound on the magnitude of $R(t)$:

$$|R(t)| < 2\varepsilon_0 \int_{\omega_0}^{\infty} C(\omega)\,\mathrm{d}\omega. \tag{10}$$

We wish to determine the asymptotic form of $r(t)$ as $t \to 0$. To this effect, introduce an arbitrary constant phase ϕ_0, and interpret $t \to 0^+$ as meaning values of t small enough to satisfy

$$0 < \omega_0 t \leqslant \phi_0 \tag{11}$$

in which case (8) expands into

$$\begin{aligned} r(t) = 2\int_0^{\omega_0} S(\omega)\cos\omega t\,\mathrm{d}\omega + 2\int_{\omega_0}^{\phi_0/t} S_{\mathrm{as}}(\omega)\cos\omega t\,\mathrm{d}\omega \\ +2\int_{\phi_0/t}^{\infty} S_{\mathrm{as}}(\omega)\cos\omega t\,\mathrm{d}\omega + R(t) \qquad (0 < t \leqslant \phi_0/\omega_0). \end{aligned} \tag{12}$$

Inserting the power series expansion of $\cos\omega t$ into the first two integrals of (12) results in

$$\begin{aligned} r(t) = 2\int_0^{\omega_0} S(\omega)\,\mathrm{d}\omega - \int_0^{\omega_0} S(\omega)\omega^2\,\mathrm{d}\omega\, t^2 + \frac{1}{12}\int_0^{\omega_0} S(\omega)\omega^4\,\mathrm{d}\omega\, t^4 + O(t^6) \\ +2\int_{\omega_0}^{\phi_0/t} S_{\mathrm{as}}(\omega)\,\mathrm{d}\omega - \int_{\omega_0}^{\phi_0/t} S_{\mathrm{as}}(\omega)\omega^2\,\mathrm{d}\omega\, t^2 + \frac{1}{12}\int_{\omega_0}^{\phi_0/t} S_{\mathrm{as}}(\omega)\omega^4\,\mathrm{d}\omega\, t^4 \\ -\frac{2}{6!}\int_{\omega_0}^{\phi_0/t} S_{\mathrm{as}}(\omega)\omega^6\,\mathrm{d}\omega\, t^6 + \cdots + 2\int_{\phi_0/t}^{\infty} S_{\mathrm{as}}(\omega)\cos\omega t\,\mathrm{d}\omega + R(t) \\ (0 < t \leqslant \phi_0/\omega_0). \end{aligned} \tag{13}$$

Equation (13) allows us to derive the asymptotic expansion of $r(t)$ as $t \to 0$ from the arbitrary conjugate asymptotic form $S(\omega)$ as $\omega \to \infty$.

5. Conjugate asymptotic forms for decaying power laws

Let us introduce the particular family of decaying power-law asymptotic forms parametrized by real, positive p:

$$S(\omega) \sim S_{\mathrm{as}}(\omega) = \frac{1}{\omega^p}(p > 0) \qquad \text{as } \omega \to \infty. \tag{14}$$

Substituting (14) into (13), and using the identity

$$\int_{\phi_0/t}^{\infty} \frac{1}{\omega^p}\cos\omega t\,\mathrm{d}\omega = \int_{\phi_0}^{\infty} \frac{1}{\phi^p}\cos\phi\,\mathrm{d}\phi\, t^{(p-1)} \tag{15}$$

we obtain the conjugate family of asymptotic expansions

$$\begin{aligned} r(t) = 2\int_0^{\omega_0} S(\omega)\,\mathrm{d}\omega - \int_0^{\omega_0} S(\omega)\omega^2\,\mathrm{d}\omega\, t^2 + \frac{1}{12}\int_0^{\omega_0} S(\omega)\omega^4\,\mathrm{d}\omega\, t^4 + O(t^6) \\ +2\int_{\omega_0}^{\phi_0/t} \omega^{-p}\,\mathrm{d}\omega - \int_{\omega_0}^{\phi_0/t} \omega^{2-p}\,\mathrm{d}\omega\, t^2 + \frac{1}{12}\int_{\omega_0}^{\phi_0/t} \omega^{4-p}\,\mathrm{d}\omega\, t^4 \\ -\frac{2}{6!}\int_{\omega_0}^{\phi_0/t} \omega^{6-p}\,\mathrm{d}\omega\, t^6 + \cdots + 2\int_{\phi_0}^{\infty} \frac{1}{\phi^p}\cos\phi\,\mathrm{d}\phi\, t^{(p-1)} + R(t) \\ \text{as } t \to 0. \end{aligned} \tag{16}$$

In the following derivations of the $t \to 0$ asymptotic approximations of $r(t)$, only the two lowest-order terms in asymptotic expansion (16) will be retained, and $C(\omega)$ will be chosen to decay to zero fast enough to ensure that $R(t)$, and any other similar remainder terms, have no influence on $r(t)$ as $t \to 0$ to this order. We shall be quite liberal in our choice of $C(\omega)$, choosing algebraic decay in preference to the more conservative exponential decay. The notion of algebraic convergence with *convergence rate* q will be understood to mean asymptotic convergence with the convergence function

$$C(\omega) = 1/\omega^q. \tag{17}$$

5.1. $0 < p < 1$

The two lowest-order terms when the $S(\omega)$ decay exponent is in the interval $0 < p < 1$ are $t^{(p-1)}$ and constant, with the t^2 and higher-order terms being neglected in (16), to yield

$$r(t) = c_0 \frac{1}{t^{(1-p)}} + c_1 + O(t^2) + R(t) \qquad \text{as } t \to 0^+ \tag{18}$$

with

$$\begin{aligned} c_0 = c_0(\phi_0) \equiv {} & \frac{2}{(1-p)}\phi_0^{(1-p)} - \frac{2}{2!(3-p)}\phi_0^{(3-p)} + \frac{2}{4!(5-p)}\phi_0^{(5-p)} \\ & - \frac{2}{6!(7-p)}\phi_0^{(7-p)} + \cdots + 2\int_{\phi_0}^{\infty} \frac{1}{\phi^p}\cos\phi\,\mathrm{d}\phi \end{aligned} \tag{19}$$

and

$$c_1 = c_1(\omega_0) \equiv 2\int_0^{\omega_0} (S(\omega) - S_{\mathrm{as}}(\omega))\,\mathrm{d}\omega. \tag{20}$$

On differentiating (19) with respect to ϕ_0, and taking into account all terms in the infinite series, we find that the right-hand side is identically zero,

$$\mathrm{d}c_0/\mathrm{d}\phi_0 = 0 \qquad \forall \phi_0 \tag{21}$$

so that c_0 is in fact independent of ϕ_0, and therefore may be evaluated from (19) at any convenient value of ϕ_0. In particular, choose $\phi_0 = 0$, so that

$$c_0 = \lim_{\phi_0 \to 0^+} c_0(\phi_0). \tag{22}$$

Applying (22) to (19) yields

$$\begin{aligned} c_0 &= 2\int_0^{\infty} \frac{1}{\phi^p}\cos\phi\,\mathrm{d}\phi \\ &= 2\Gamma(1-p)\sin\left(p\frac{\pi}{2}\right) \end{aligned} \tag{23}$$

where $\Gamma(x)$ is the Euler gamma function [18].

Subject to the existence of the introduced improper integral, (20) may be expressed as

$$c_1(\omega_0) = 2\int_0^{\infty} (S(\omega) - S_{\mathrm{as}}(\omega))\,\mathrm{d}\omega - R(0) \tag{24}$$

where an upper bound on the magnitude of the remainder term $-R(0)$ is as in (10).

To progress further we have to choose a suitable convergence rate q, to completely specify the convergence function $C(\omega)$ according to (17). We soon realize that a suitable convergence rate must be strictly greater than one, that is,

$$q = 1 + \delta \qquad \delta > 0. \tag{25}$$

Now we have all the information required to explicitly evaluate bound (10), yielding

$$|R(t)| < \frac{2\varepsilon_0}{\delta}\frac{1}{\omega_0^{\delta}}. \tag{26}$$

Substituting (23) and (24) into asymptotic expansion (18) gives

$$r(t) = 2\Gamma(1-p)\sin\left(p\frac{\pi}{2}\right)\frac{1}{t^{(1-p)}} + 2\int_0^{\infty}(S(\omega) - S_{\mathrm{as}}(\omega))\,\mathrm{d}\omega$$
$$+O(t^2) + (R(t) - R(0)) \qquad \text{as } t \to 0^+ \tag{27}$$

where a bound on the combined remainder term is computable from (26), being

$$|R(t) - R(0)| < \frac{4\varepsilon_0}{\delta}\frac{1}{\omega_0^{\delta}}. \tag{28}$$

From (11), $\omega_0 \leqslant \phi_0/t$, so that as $t \to 0^+$, ω_0 can be made arbitrarily large. This in turn allows ε_0 to become arbitrarily small as befits asymptotic convergence. From (28), the combination of these two influences is that $|R(t) - R(0)|$ becomes arbitrarily small. Obviously $O(t^2)$ also becomes arbitrarily small in the same situation. At the same time, as $t \to 0^+$ the first term on the right-hand side of (27) increases without bound, and the second term remains a constant that is non-zero in general. Accordingly, for a convergence rate $q > 1$, the influence of remainder terms on the two lowest-order terms of the asymptotic expansion of $r(t)$ as $t \to 0^+$ does indeed vanish, leaving the desired result as

$$r(t) = 2\Gamma(1-p)\sin\left(p\frac{\pi}{2}\right)\frac{1}{|t|^{(1-p)}} + 2\int_0^{\infty}(S(\omega) - S_{\mathrm{as}}(\omega))\,\mathrm{d}\omega \qquad \text{as } t \to 0. \tag{29}$$

Let us reflect upon the accessibility of asymptotic limit (29); that is, how closely t must approach zero before $r(t)$ is accurately described by (29). The only difficulty with the preceding argument occurs if δ is very small, that is, q is very close to one. In this case, bound (28) requires ω_0 to be extremely large and ε_0 to be extremely small, before it becomes negligibly small. However, these constraints on ω_0 and ε_0 are achieved only at extremely small t. Thus, remainder term $(R(t) - R(0))$ in (27) becomes negligible compared with the two lowest-order terms only as t approaches extremely close to zero. We summarize this argument by stating that asymptotic approximation (29) is easily accessible for the parameter intervals $0 < p < 1$ and $q > 1+$, where $x+$ $(x-)$ is interpreted as meaning x plus (minus) a small, but finite, number.

5.2. $p = 1$

The two lowest-order terms when the $S(\omega)$ decay exponent is $p = 1$ are $\ln t$ and constant, with the t^2 and higher-order terms being neglected in (16), which becomes

$$r(t) = -2\ln t + c_2 + O(t^2) + R(t) \qquad \text{as } t \to 0^+. \tag{30}$$

Similarly to c_0 in section 5.1, c_2 is defined as a function of both ω_0 and ϕ_0, which turns out to be constant with respect to ϕ_0, so is calculable at any ϕ_0, in particular $\phi_0 \to 0^+$, giving

$$c_2 = c_2(\omega_0) = 2\int_0^{\omega_0} S(\omega)\,\mathrm{d}\omega - 2\ln\omega_0 + \lim_{\phi_0\to 0^+}\left(2\ln\phi_0 + 2\int_{\phi_0}^{\infty}\frac{1}{\phi}\cos\phi\,\mathrm{d}\phi\right)$$
$$= 2\int_0^{\omega_0} S(\omega)\,\mathrm{d}\omega - 2\ln\omega_0 - 2\gamma \tag{31}$$

where $\gamma \approx 0.577\,216$ is Euler's constant [18].

Subject to the existence of the introduced improper integral, (31) may be expressed as

$$c_2(\omega_0) = 2\int_0^1 S(\omega)\,\mathrm{d}\omega + 2\int_1^\infty (S(\omega) - S_{\mathrm{as}}(\omega))\,\mathrm{d}\omega - 2\gamma - R(0). \tag{32}$$

We impose a convergence rate $q > 1$ as in section 5.1, and an identical analysis once again demonstrates that the remainder terms become negligible in the small t asymptotic limit, in which case asymptotic expansion (30) is approximated to second order by

$$r(t) = -2\ln|t| + 2\int_0^1 S(\omega)\,\mathrm{d}\omega + 2\int_1^\infty (S(\omega) - S_{\mathrm{as}}(\omega))\,\mathrm{d}\omega - 2\gamma \qquad \text{as } t \to 0. \tag{33}$$

For the same reason as enunciated in section 5.1, asymptotic approximation (33) is easily accessible only if $q > 1+$.

5.3. $1 < p < 3$

The two lowest-order terms of the asymptotic expansion when the $S(\omega)$ decay exponent is in the interval $1 < p < 3$, are constant and $t^{(p-1)}$, with the t^2 and higher-order terms being neglected in (16), to yield

$$r(t) = c_3 + c_0 t^{(p-1)} + O(t^2) + R(t) \qquad \text{as } t \to 0^+ \tag{34}$$

with

$$c_3 = c_3(\omega_0) \equiv 2\int_0^{\omega_0} S(\omega)\,\mathrm{d}\omega + 2\int_{\omega_0}^\infty S_{\mathrm{as}}(\omega)\,\mathrm{d}\omega \tag{35}$$

and $c_0(\phi_0)$ being exactly as defined in (19).

For the same reason as cited in section 5.1, c_0 is still given by (22), but now the result of this limit is

$$\begin{aligned} c_0 &= \lim_{\phi_0 \to 0^+} \frac{2}{(1-p)}\frac{1}{\phi_0^{(p-1)}} + 2\int_{\phi_0}^\infty \frac{1}{\phi^p}\cos\phi\,\mathrm{d}\phi \\ &= 2\Gamma(1-p)\sin\left(p\frac{\pi}{2}\right) \\ &= -\pi \qquad \text{for } p = 2. \end{aligned} \tag{36}$$

Subject to the existence of the introduced improper integral, (35) may be expressed as

$$c_3(\omega_0) = 2\int_0^\infty S(\omega)\,\mathrm{d}\omega - R(0). \tag{37}$$

The convergence rate q will be chosen as in (25), whereupon the remainder terms bound in (26) is regained.

Substituting (37) and (36) into (34) gives

$$\begin{aligned} r(t) = 2\int_0^\infty S(\omega)\,\mathrm{d}\omega &+ 2\Gamma(1-p)\sin\left(p\frac{\pi}{2}\right) t^{(p-1)} \\ &+ O(t^2) + (R(t) - R(0)) \qquad \text{as } t \to 0^+ \end{aligned} \tag{38}$$

subject to bound (28). We draw assurance from the fact that (38) implies

$$\lim_{t \to 0^+} r(t) = r(0) = 2\int_0^\infty S(\omega)\,\mathrm{d}\omega \tag{39}$$

as it should for possibly integrable $S(\omega)$ (i.e. finite power signals). Note that for the previous cases of $0 < p \leqslant 1$, $\lim_{t\to 0^+} r(t)$ does not exist, since $r(t) \to \infty$ as $t \to 0^+$, reflecting the definite non-integrability of $S(\omega)$ (i.e. infinite power signals). Substituting (39) into (38) yields

$$r(t) = r(0) + 2\Gamma(1-p)\sin\left(p\frac{\pi}{2}\right)t^{(p-1)} + O(t^2) + (R(t) - R(0)) \qquad \text{as } t \to 0^+. \quad (40)$$

For the same reason as cited in section 5.1, ε_0 can be made arbitrarily small as t approaches arbitrarily close to zero. Furthermore, from (11),

$$\omega_0 \leqslant \phi_0/t \Rightarrow \frac{1}{\omega_0^\delta} = \frac{B}{\phi_0^\delta}t^\delta \qquad \text{where } B \geqslant 1. \quad (41)$$

By choosing ω_0 sufficiently close to ϕ_0/t, B can be made arbitrarily close to one—the important property is that B can be made a finite number. Substitution of (41) into (28) yields

$$|R(t) - R(0)| < \frac{4\varepsilon_0(t)B}{\delta\phi_0^\delta}t^\delta = o(t^\delta) \qquad \text{as } t \to 0^+ \quad (42)$$

where the property that the allowable smallness of ε_0 depends on the chosen smallness of t is indicated explicitly by writing $\varepsilon_0(t)$; and where expressing the bound as an $o(\cdot)$ is justified by the fact that the coefficient of t^δ has only finite factors, apart from $\varepsilon_0(t)$, which approaches zero as $t \to 0^+$.

It automatically holds that

$$O(t^2) = o(t^{(p-1)}) \qquad \text{as } t \to 0^+ \quad (43)$$

and if we choose $\delta \geqslant (p-1)$, then

$$o(t^\delta) = o(t^{(p-1)}) \qquad \text{as } t \to 0^+ \text{ for } \delta \geqslant (p-1). \quad (44)$$

Equivalently from (25), for $q \geqslant p$, (42) and (44) combine to give

$$|R(t) - R(0)| = o(t^{(p-1)}) \qquad \text{as } t \to 0^+ \text{ for } q \geqslant p. \quad (45)$$

As $t \to 0^+$, the first two terms on the right-hand side of (40) are $O(1)$ and $O(t^{(p-1)})$, respectively, so they comprehensively dominate the remainder terms in this limit. Therefore, to the required accuracy the desired asymptotic approximation is

$$r(t) = r(0) + 2\Gamma(1-p)\sin\left(p\frac{\pi}{2}\right)|t|^{(p-1)} \qquad \text{as } t \to 0 \quad (46)$$

this expansion being exact at $t = 0$.

Turning our attention to the accessibility of asymptotic limit (46), we note that there are two conditions under which (46) is achieved only at exceptionally small t. One condition is when δ is extremely small, or equivalently q is extremely close to one. In this situation, the middle expression in (42) becomes much smaller than t^δ only when $\varepsilon_0(t)$ becomes exceptionally small, which requires ω_0 to be chosen exceptionally large, which in turn requires t to be exceptionally small, since $t \leqslant \phi_0/\omega_0$. The other condition is when p is extremely close to three. In this situation the difficulty is with (43), since t^2 becomes much smaller than $t^{(p-1)}$ only when t is exceptionally small. In our notation this reasoning implies that asymptotic approximation (46) is easily accessible for the parameter intervals $1 < p < 3-$ and $q \geqslant \sup(p, 1+)$.

5.4. $p = 3$

The two lowest-order terms of the asymptotic expansion when the $S(\omega)$ decay exponent is $p = 3$, are constant and $t^2 \ln t$, with the t^2 and higher-order terms being neglected in (16), which becomes

$$r(t) = c_3 + t^2 \ln t + O(t^2) + R(t) \qquad \text{as } t \to 0^+ \tag{47}$$

with $c_3(\omega_0)$ defined as in (35), and simplifying to (37). Substituting (37) into (47) we obtain

$$r(t) = 2\int_0^\infty S(\omega)\,\mathrm{d}\omega + t^2 \ln t + O(t^2) + (R(t) - R(0)) \qquad \text{as } t \to 0^+ \tag{48}$$

with a bound on $(R(t) - R(0))$ given by (42). Once again, (48) implies (39), consistent with the possible integrability of $S(\omega)$, which, when substituted into (48), gives

$$r(t) = r(0) + t^2 \ln t + O(t^2) + (R(t) - R(0)) \qquad \text{as } t \to 0^+. \tag{49}$$

It is true that

$$O(t^2) = o(t^2 \ln t) \qquad \text{as } t \to 0^+ \tag{50}$$

and if we choose $\delta \geqslant 2$ in (25), or equivalently $q \geqslant 3$, then it is also true that (42) becomes

$$|R(t) - R(0)| = o(t^\delta) = o(t^2 \ln t) \qquad \text{as } t \to 0^+ \text{ for } q \geqslant 3. \tag{51}$$

Recognizing that

$$t^2 \ln t = o(1) \qquad \text{as } t \to 0^+ \tag{52}$$

we see that all of the remainder terms in asymptotic expansion (49) are negligible compared with the two lowest-order terms, and that the desired asymptotic approximation to the required accuracy is

$$r(t) = r(0) + t^2 \ln |t| \qquad \text{as } t \to 0 \tag{53}$$

this expansion being exact at $t = 0$.

Because $\ln t$ has a weak singularity at $t = 0$, t must be exceptionally small before the $O(t^2)$ remainder term becomes insignificant relative to the $t^2 \ln t$ term. The same situation arises with the $(R(t) - R(0))$ remainder term, for q too close to three. Accordingly, asymptotic approximation (53) becomes accurate only at exceptionally small t, or in our introduced terminology, it has poor accessibility.

5.5. $3 < p < 5$

When the $S(\omega)$ decay exponent is in the interval $3 < p < 5$, the two lowest-order terms of asymptotic expansion (16) are constant and t^2, and neglecting the $t^{(p-1)}$ and higher-order terms, we obtain

$$r(t) = c_3 + c_4 t^2 + O(t^{(p-1)}) + R(t) \qquad \text{as } t \to 0^+ \tag{54}$$

where $c_3(\omega_0)$ is defined by (35), and

$$c_4 = c_4(\omega_0) \equiv -\int_0^{\omega_0} S(\omega)\omega^2\,\mathrm{d}\omega - \int_{\omega_0}^\infty S_{\mathrm{as}}(\omega)\omega^2\,\mathrm{d}\omega. \tag{55}$$

$c_3(\omega_0)$ simplifies to (37), and subject to the existence of the introduced improper integral, $c_4(\omega_0)$ simplifies to

$$c_4(\omega_0) = -\int_0^\infty S(\omega)\omega^2\,\mathrm{d}\omega - \frac{1}{2}R''(0) \tag{56}$$

where primes indicate differentiation. The asymptotic convergence criterion of section 3 induces the remainder term upper bound

$$\left|\frac{1}{2}R''(0)\right| < \varepsilon_0 \int_{\omega_0}^{\infty} C(\omega)\omega^2 \,\mathrm{d}\omega. \tag{57}$$

Substituting (37) and (56) into (54) derives

$$r(t) = 2\int_0^\infty S(\omega)\,\mathrm{d}\omega - \int_0^\infty S(\omega)\omega^2\,\mathrm{d}\omega\, t^2 \\ +O(t^{(p-1)}) + (R(t) - R(0)) - \frac{1}{2}R''(0)t^2 \qquad \text{as } t \to 0^+. \tag{58}$$

We see that (58) infers both (39) and

$$\lim_{t\to 0^+} r''(t) = r''(0) = -2\int_0^\infty S(\omega)\omega^2\,\mathrm{d}\omega \tag{59}$$

as expected for possibly integrable $S(\omega)$ and $S(\omega)\omega^2$. On substitution of (39) and (59), (58) simplifies to

$$r(t) = r(0) + \tfrac{1}{2}r''(0)t^2 + O(t^{(p-1)}) + (R(t) - R(0)) - \tfrac{1}{2}R''(0)t^2 \qquad \text{as } t \to 0^+. \tag{60}$$

We now choose a suitable convergence rate q. It transpires that the convergence rate must be greater than three to provide assurance of the insignificance of all remainder terms in (60). Consequently, we impose the condition $\delta > 2$ in (25) to ensure that $q > 3$. A bound on $(R(t) - R(0))$ is as specified in (42), and an analogous derivation to that in sections 5.1 and 5.3 proceeds from (57) to the following bound on the $1/2R''(0)t^2$ remainder term

$$\left|\frac{1}{2}R''(0)t^2\right| < \frac{\varepsilon_0}{(\delta-2)}\frac{1}{\omega_0^{(\delta-2)}}t^2 = \frac{\varepsilon_0(t)D}{(\delta-2)\phi_0^{(\delta-2)}}t^\delta = o(t^\delta) \qquad \text{as } t \to 0^+ \tag{61}$$

where $D \geqslant 1$ can be made a finite number. Had we chosen a convergence rate less than or equal to three, we would have obtained an infinite 'bound' on $R''(0)$, so that $R''(0)t^2$ would become strictly greater than $O(t^2)$ as $t \to 0^+$—certainly not negligible compared with the explicitly retained t^2 term in the asymptotic expansion. Comparing (42) and (61), we notice that although the bounds that we have identified for $(R(t) - R(0))$ and $1/2R''(0)t^2$, respectively, are numerically different, the important feature is that they have the same order, being $o(t^\delta)$ as $t \to 0^+$.

We note that

$$O(t^{(p-1)}) = o(t^2) \qquad \text{as } t \to 0^+ \tag{62}$$

and that for $\delta > 2$ (i.e. $q > 3$),

$$o(t^\delta) = o(t^2) \qquad \text{as } t \to 0^+ \text{ for } \delta > 2. \tag{63}$$

Substitutiing (63) into (42) and (61) yields

$$|R(t) - R(0)|, |R''(0)t^2| = o(t^2) \qquad \text{as } t \to 0^+ \text{ for } q > 3. \tag{64}$$

The two lowest-order terms in asymptotic expansion (60) are $O(1)$ and $O(t^2)$ as $t \to 0^+$, so all of the remainder terms, being $o(t^2)$ as $t \to 0^+$, are suitably negligible in the small t limit for the correct second-order asymptotic approximation to be

$$r(t) = r(0) + \tfrac{1}{2}r''(0)\,t^2 \qquad \text{as } t \to 0 \tag{65}$$

this expansion being exact at $t = 0$.

As in section 5.3, closer examination of the foregoing analysis reveals that in the two circumstances of q being extremely close to three (i.e. δ extremely close to 2), or p being very close to three, asymptotic limit (65) is achieved only as t becomes extremely small. In our notation, asymptotic limit (65) is easily accessible for the parameter ranges $3+ < p < 5$ and $q > 3+$.

5.6. $p = 5$

The $S(\omega)$ decay exponent $p = 5$ asymptotic approximation, and its derivation, is identical to the $3 < p < 5$ case considered in section 5.5, except that the $O(t^{(p-1)})$ remainder term in (54) is replaced by an $O(t^4 \ln t)$ remainder term. Since

$$O(t^4 \ln t) = o(t^2) \qquad \text{as } t \to 0^+ \tag{66}$$

asymptotic approximation (65) is still applicable, with the same condition on the convergence rate as enunciated in section 5.5, and the same accessibility situation still prevailing.

5.7. $p > 5$

Likewise, the $p > 5$ case also is identical to the $3 < p < 5$ case considered in section 5.5, except that the $O(t^{(p-1)})$ remainder term is now replaced by an $O(t^4)$ remainder term. Since

$$O(t^4) = o(t^2) \qquad \text{as } t \to 0^+ \tag{67}$$

asymptotic approximation (65) is still applicable, and so are the corresponding condition on convergence rate, and accessibility situation.

6. Conditions on convergence rates

Note that the upper bounds on remainder terms that we deduced in the derivation of sections 4 and 5 are not supremums (i.e. least upper bounds). So, in general, the actual remainders will be smaller than our bounds for them. However, the conditions that we imposed on the convergence rates q, to ensure validity of our derived small t asymptotic approximations, were dictated by our bounds for the remainder terms. If we could have identified tighter bounds for the remainder terms, then there would have been a commensurate relaxation in the conditions imposed on q. Therefore, we realize that the conditions imposed on q throughout section 5 are sufficient, but not necessary, for the derived asymptotic approximations to be valid.

7. Significance of conjugate asymptotic properties

Let us consider the utility of the conjugate asymptotic behaviours derived (partially, but generally) in section 4 and (completely, but specifically) in section 5. In section 1 we identified the specific research problem of Yang and Lu [6] as potentially being a particularly appropriate beneficiary of our results. Now we shall explore more general prospective applications.

A natural phenomenon that is associated with a stochastic process is partially characterized by its correlation in time space, or its spectrum in frequency space. For some phenomena, experiment indicates that either the correlation or the spectrum decays with a certain functional dependence on its argument as the argument becomes large. This behaviour would be observed up to a limiting value of the argument; for correlations the time lag must be shorter than the time interval over which signals are measured, for spectra the frequency is limited by the spectral range of the spectrometer, and for both the respective decays cannot be followed beneath the measuring instrument noise threshold. Although the characteristic decay is observed only over a finite domain, often a mathematically appealing and scientifically defensible assumption is that the noted characteristic decay continues *ad infinitum*. For other phenomena, belief in the intrinsic simplicity and symmetry (e.g. invariance with respect to scaling) of nature may dictate that the decay in the correlation or spectrum at large argument values continues with its characteristic form all the way to infinity in its domain.

In some circumstances it is imperative that the conjugate small argument behaviour be determined, because that may be the very quantity that a proposed physical theory for the phenomenon calculates most reliably and easily. For example, statistical mechanics with reference to atomic and molecular theory would be used to directly calculate the pair correlation function for non-ideal gases, liquids or solids, but x-ray or neutron diffraction experiments would actually measure the associated spectrum over a finite wavenumber domain. Numerically evaluating the Fourier transform of the measured spectrum, even an extrapolated version thereof, will not necessarily yield an accurate approximation for the pair correlation near the origin, because such a computation will effectively truncate the large wavenumber decay at some finite value, due to time and precision constraints. The definitive functional form being sought is only revealed by analysis, such as the one presented here. Only then do we have an accurate experimental determination of the pair correlation function very close to the origin, to compare with the theoretical determination of such.

In summary, the conjugate asymptotic properties of correlations and spectra allow extrapolations to infinity in one space to dictate the extrapolation to zero in the conjugate space, without the need for possibly intractable numerical computations. This may enable comparison between experiment and theory for the natural phenomenon under investigation.

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